

KAKUTANI DICHOTOMY ON FREE STATES

TAKU MATSUI AND SHIGERU YAMAGAMI*

ABSTRACT. Two quasi-free states on a CAR or CCR algebra are shown to generate quasi-equivalent representations unless they are disjoint.

1. INTRODUCTION

Kakutani's celebrated dichotomy theorem on infinite product measures opened a way to mathematical analysis in infinite dimensional phenomena. In classical probability theory, lots of related results have been explored since then, whereas in quantum probability, this has been mostly done with relations to infinite tensor products of states of quantum algebras. Especially quasi-free states of so-called CAR algebras and CCR algebras were investigated much around 1970's from the view point of equivalence of representations and explicit criteria for their quasi-equivalence are obtained in terms of Hilbert-Schmidt class operators.

In this paper, we shall add a complement to this old subject by establishing dichotomies on quasi-free states: Given quasi-free states φ and ψ of a CAR or CCR algebra, one of the following alternatives occurs.

- (i) φ and ψ are quasi-equivalent.
- (ii) φ and ψ are disjoint.

In the case of CCR algebras, these alternatives are further related with non-vanishing or vanishing of transition probabilities between quasi-free states, which therefore inherits the same spirit with the original dichotomy due to S. Kakutani.

2. PRELIMINARIES

We shall freely use the standard terminologies in operator algebras and the notations introduced in [15] with some of basic ones repeated here for the reader's convenience. Given a C*-algebra C , $L^2(C)$ denotes the standard Hilbert space of the enveloping von Neumann algebra C^{**}

*Partially supported by KAKENHI(22540217).

with the natural left and right actions of C on $L^2(C)$. For a state φ of C , the realizing vector in the positive cone of $L^2(C)$ is denoted by $\varphi^{1/2}$. The projection to the closed subspace $\overline{C\varphi^{1/2}C} \subset L^2(C)$ is then equal to the central support of φ , which is a projection in the center of C^{**} . As a consequence, two states φ and ψ produce quasi-equivalent GNS representations if and only if $\overline{C\varphi^{1/2}C} = \overline{C\psi^{1/2}C}$, whereas they are disjoint if and only if $\overline{C\varphi^{1/2}C} \perp \overline{C\psi^{1/2}C}$.

In this framework, we have several possibilities for transition probability between states. Most known is the Uhlmann's one, which is the square of the so-called fidelity $\rho(\varphi, \psi)$ between states φ, ψ (see [1] for further information). In our context of non-commutative L^p -theory (see [5] among several approaches to the subject and also cf. [14]), $\rho(\varphi, \psi)$ is equal to the norm of the positive linear functional $|\varphi^{1/2}\psi^{1/2}| = \sqrt{\varphi^{1/2}\psi\varphi^{1/2}}$ in C^* ([10]). Another choice is $(\varphi^{1/2}|\psi^{1/2})$, which is reduced to the ordinary transition probability for vector states on $\mathcal{B}(\mathcal{H})$ and will play similar roles as Hellinger integrals did in the Kakutani's dichotomy theorem ([6]). Thus its vanishing or non-vanishing is our main concern here and the fidelity can be equally well used for this purpose in view of inequalities $(\varphi^{1/2}|\psi^{1/2})^2 \leq \rho(\varphi, \psi)^2 \leq (\varphi^{1/2}|\psi^{1/2})$.

For free states of quantum algebras, we know decisive results for the criterion of quasi-equivalence and the closed formula of transition probability. To explain these, we recall relevant definitions.

Given a real Hilbert space V with inner product (x, y) ($x, y \in V$), the CAR algebra is a unital C^* -algebra $C(V)$ linearly generated by elements of V with the relations

$$x^* = x, \quad xy + yx = (x, y)1, \quad x, y \in V.$$

Likewise, given a real vector space V and an alternating bilinear form σ on V , the CCR C^* -algebra is the C^* -algebra $C(V, \sigma)$ generated universally by the symbols $\{e^{ix}\}_{x \in V}$ with the relations

$$(e^{ix})^* = e^{-ix}, \quad e^{ix}e^{iy} = e^{-i\sigma(x, y)/2}e^{i(x+y)}, \quad x, y \in V.$$

Remark that we allow σ to be degenerate, whence our CCR C^* -algebras may have non-trivial centers.

Given a state φ of a CAR algebra $C(V)$, the covariance operator S on the complexified Hilbert space $V^{\mathbb{C}}$ is defined by $\varphi(x^*y) = (x, Sy)$, which turns out to be positive and satisfies the relation $S + \overline{S} = I$, where \overline{S} is the complex conjugate of S and I denotes the identity operator. A state is said to be quasi-free and denoted by φ_S if it vanishes on the

odd part of $C(V)$ and satisfies the recursive relation

$$\begin{aligned} \varphi(x_1 x_2 \dots x_{2n}) &= \varphi(x_1 x_2) \varphi(x_3 x_4 \dots x_{2n}) \\ &\quad - \varphi(x_1 x_3) \varphi(x_2 x_4 \dots x_{2n}) + \dots + \varphi(x_1 x_{2n}) \varphi(x_2 \dots x_{2n-1}). \end{aligned}$$

If the recursive computations are worked out completely, the Wick formula is obtained:

$$\varphi(x_1 x_2 \dots x_{2n}) = \sum \pm \prod_{k=1}^n \varphi(x_{i_k} x_{j_k}),$$

where the summation is taken over all the way of pairings in $\{1, 2, \dots, 2n\}$ and \pm is chosen according to the signature of the permutation sequence $(i_1, j_1, \dots, i_n, j_n)$.

In the case of CCR C^* -algebra, a state φ is said to be quasi-free and denoted by φ_S if

$$\varphi(e^{ix}) = e^{-S(x,x)/2},$$

where S is a positive sesqui-linear form on the complexified vector space $V^{\mathbb{C}}$ and is referred to as the covariance form of φ . We know that a positive form S on $V^{\mathbb{C}}$ is a covariance form if and only if

$$S(x, y) - \overline{S}(x, y) = i\sigma(x, y)$$

for $x, y \in V^{\mathbb{C}}$. Here $\overline{S}(x, y) = \overline{S(\overline{x}, \overline{y})}$ and σ is sesqui-linearly extended to $V^{\mathbb{C}}$.

Given a quasi-free state φ_S , we write $L^2(S) = \overline{C\varphi_S^{1/2}C}$ with $C = C(V)$ (CAR case) or $C = C(V, \sigma)$ (CCR case). Notice here that the same letter is used to stand for a covariance operator or a covariance form according to the case of CAR or CCR.

For quasi-free states, quasi-equivalence criteria were investigated by many researchers but let us just indicate [2], [8], [11], [12] and [13] among them. The following form is due to [2] and [3].

Theorem 2.1 (Quasi-Equivalence Criteria).

- (i) Let φ_S and φ_T be quasi-free states of a CAR algebra with covariance operators S and T . Then φ_S and φ_T are quasi-equivalent if and only if $\sqrt{S} - \sqrt{T}$ is in the Hilbert-Schmidt class.
- (ii) Let φ_S and φ_T be quasi-free states of a CCR C^* -algebra with covariance forms S and T . Then φ_S and φ_T are quasi-equivalent if and only if $S + \overline{S} \cong T + \overline{T}$ as inner products and $\sqrt{\frac{S}{S + \overline{S}}} - \sqrt{\frac{T}{T + \overline{T}}}$ is in the Hilbert-Schmidt class.

The following determinant formulas for transition probabilities are due to [2] (CAR case) and [16] (CCR case).

Theorem 2.2 (Transition Probability Formula).

(i) Let S and T be covariance operators for a CAR algebra. Then

$$(\varphi_S^{1/2} | \varphi_T^{1/2})^4 = \det(MM^*),$$

where $M = S^{1/2}T^{1/2} + (I - S)^{1/2}(I - T)^{1/2}$.

(ii) Let S and T be covariance forms for a CCR C^* -algebra. Then

$$(\varphi_S^{1/2} | \varphi_T^{1/2})^2 = \det \left(\frac{2\sqrt{AB}}{A+B} \right),$$

where positive forms A and B are defined by

$$2A = S + 2\sqrt{S\bar{S}} + \bar{S}, \quad 2B = T + 2\sqrt{T\bar{T}} + \bar{T},$$

and their geometric mean \sqrt{AB} as well as $\sqrt{S\bar{S}}$ and $\sqrt{T\bar{T}}$ is in the sense of [9].

3. CAR DICHOTOMY

Let ϵ be the parity automorphism of $C(V)$ and define a bounded linear operator $\pi(\xi \oplus \eta)$ on $L^2(C(V))$ by

$$\pi(\xi \oplus \eta)\psi^{1/2} = \xi\psi^{1/2} + (\psi \circ \epsilon)^{1/2}\eta.$$

Here $\xi, \eta \in V$ and ψ is a state of $C(V)$. Then

$$\pi(\bar{\xi} \oplus -\bar{\eta})\pi(\xi' \oplus \eta') + \pi(\xi' \oplus \eta')\pi(\bar{\xi} \oplus -\bar{\eta}) = (\xi|\xi') + (\eta|\eta')$$

and π is extended to a $*$ -representation of $C(V \oplus iV)$, which is referred to as the **quadrature representation** of $C(V \oplus iV)$. Here iV denotes the real part of $V^{\mathbb{C}}$ with respect to the conjugation given by $x \mapsto -\bar{x}$. Note that, if ψ is an even state, i.e., $\psi \circ \epsilon = \psi$,

$$\pi(C(V \oplus iV))\psi^{1/2} = C(V)\psi^{1/2}C(V).$$

In particular, for a quasi-free state φ_S of covariance operator S , π leaves the closed central subspace $L^2(S) = C(V)\varphi_S^{1/2}C(V)$ invariant. Let π_S be the associated subrepresentation of $C(V \oplus iV)$.

We define the **quadrature** of a state φ of $C(V)$ to be a state Φ of $C(V \oplus iV)$ given by

$$\Phi(x) = (\varphi^{1/2} | \pi(x)\varphi^{1/2}), \quad x \in C(V \oplus iV).$$

The following is well-known (see [2] for example).

Lemma 3.1. The following conditions on a covariance operator S are equivalent.

- (i) $\ker S = \{0\}$.
- (ii) $\ker(I - S) = \{0\}$.
- (iii) $S = (1 + e^H)^{-1}$ with H a self-adjoint operator on $V^{\mathbb{C}}$ satisfying $\overline{H} = -H$.

A covariance operator S is said to be **non-degenerate** if it satisfies these equivalent conditions.

Given a covariance operator S on $V^{\mathbb{C}}$, its quadrature is defined to be the projection

$$P = \begin{pmatrix} S & \sqrt{S(I - S)} \\ \sqrt{S(I - S)} & I - S \end{pmatrix}$$

on $V^{\mathbb{C}} \oplus V^{\mathbb{C}}$, which is a covariance operator for the real Hilbert space $V \oplus iV$.

Proposition 3.2. The quadrature of φ_S is equal to the Fock state φ_P . In particular, the representation π_S is irreducible.

Proof. Recall that the Fock vacuum $\varphi_P^{1/2}$ is characterized by the vanishing property under the left multiplication of the range of \overline{P} . Since the range of \overline{P} is equal to $\{\sqrt{I - S}\zeta \oplus -\sqrt{S}\zeta; \zeta \in V^{\mathbb{C}}\}$, it suffices to show that

$$(\sqrt{I - S}\zeta)\varphi_S^{1/2} = \varphi_S^{1/2}(\sqrt{S}\zeta) \quad \text{for } \zeta \in V^{\mathbb{C}}.$$

If S is non-degenerate, this follows from the fact that φ_S is a KMS-state with respect to the one-parameter automorphism group induced from the Bogoliubov transformations $\{e^{itH}\}_{t \in \mathbb{R}}$ (see [4, Example 5.3.24] for example).

To deal with the degenerate case, let E be the projection to $\ker S(I - S)$ and write $(I - E)V^{\mathbb{C}} = W^{\mathbb{C}}$ with W a closed real subspace of V . Let φ_W (resp. ψ) be the restriction of φ_S to the C^* -subalgebra $C(W) \subset C(V)$ (resp. the C^* -subalgebra $C(W^{\perp}) \subset C(V)$), which is a quasi-free state of the reduced covariance operator $S(I - E)$ (resp. SE). Let u be the unitary operator on the Fock space $\overline{C(W^{\perp})\psi^{1/2}}$ defined by

$$u(\eta_1 \cdots \eta_n \psi^{1/2}) = (-1)^n \eta_1 \cdots \eta_n \psi^{1/2} \quad \text{for } \eta_1, \dots, \eta_n \in W^{\perp},$$

which implements the parity automorphism of $C(W^{\perp})$.

A representation θ of $C(V)$ on $\overline{C(W)\varphi_W^{1/2}} \otimes \overline{C(W^{\perp})\psi^{1/2}}$ is then defined by the correspondance

$$\xi + \eta \mapsto \xi \otimes u + 1 \otimes \eta, \quad \xi \in W, \eta \in W^{\perp}$$

on generators, where ξ and η on the right side denote operators by left multiplication. From $u\psi^{1/2} = \psi^{1/2}$ and the Wick formula, we have the equality

$$\begin{aligned} & (\varphi_W^{1/2} \otimes \psi^{1/2} | (\xi_1 \cdots \xi_m \otimes u^m \eta_1 \cdots \eta_n) (\varphi_W^{1/2} \otimes \psi^{1/2})) \\ &= \varphi_W(\xi_1 \cdots \xi_m) \psi(\eta_1 \cdots \eta_n) = \varphi_S(\xi_1 \cdots \xi_m \eta_1 \cdots \eta_n), \end{aligned}$$

which implies that $\xi_1 \cdots \xi_m \eta_1 \cdots \eta_n \varphi_S^{1/2} \mapsto \phi(\xi_1 \cdots \xi_m \eta_1 \cdots \eta_n) (\varphi_W^{1/2} \otimes \psi^{1/2})$ gives rise to an isometry U . Since the operator u is approximated by elements in $C(W^\perp)$ on $\overline{C(W^\perp)\psi^{1/2}}$ thanks to the irreducibility of representation, U is in fact surjective and θ is extended to an isomorphism $C(V)'' \rightarrow C(W)'' \otimes \mathcal{B}(\overline{C(W^\perp)\psi^{1/2}})$ of von Neumann algebras so that $\varphi_S = (\varphi_W \otimes \psi)\theta$, which in turn induces an isometric isomorphism

$$\Theta : \overline{C(V)\varphi_S^{1/2}C(V)} \rightarrow \overline{C(W)\varphi_W^{1/2}C(W)} \otimes \overline{C(W^\perp)\psi^{1/2}C(W^\perp)}$$

by the relation

$$\Theta(x\varphi_S^{1/2}x') = \theta(x)(\varphi_W^{1/2} \otimes \psi^{1/2})\theta(x'), \quad x, x' \in C(V).$$

Now, for $\xi + \eta \in V^\mathbb{C} = (I - E)V^\mathbb{C} + EV^\mathbb{C}$, in view of $((I - S)\eta)\psi^{1/2} = 0 = \psi^{1/2}(S\eta)$, we see that

$$\begin{aligned} \Theta(\varphi_S^{1/2}(\sqrt{S}(\xi + \eta))) &= (\varphi_W^{1/2} \otimes \psi^{1/2})\theta(\sqrt{S}(\xi + \eta)) \\ &= (\varphi_W^{1/2} \otimes \psi^{1/2})(\sqrt{S}\xi \otimes u + 1 \otimes \sqrt{S}\eta) \\ &= \varphi_W^{1/2}(\sqrt{S}\xi) \otimes \psi^{1/2} = (\sqrt{I - S}\xi)\varphi_W^{1/2} \otimes \psi^{1/2} \\ &= \theta(\sqrt{I - S}(\xi + \eta))(\varphi_W^{1/2} \otimes \psi^{1/2}) \\ &= \Theta(\sqrt{I - S}(\xi + \eta)\varphi_S^{1/2}). \end{aligned}$$

□

Theorem 3.3 (Dichotomy). Let S and T be covariance operators for a CAR algebra $C(V)$ with P and Q their quadratures. Let $L^2(S) = \overline{C(V)\varphi_S^{1/2}C(V)}$ and similarly for $L^2(T)$. Then $L^2(S) \perp L^2(T)$ unless $L^2(S) = L^2(T)$. Moreover, we have

$$(\varphi_P^{1/2} | \varphi_Q^{1/2}) = (\varphi_S^{1/2} | \varphi_T^{1/2})^2.$$

Proof. Since π is irreducible on both of $L^2(S)$ and $L^2(T)$, they are either unitarily equivalent or disjoint as representations of $C(V \oplus iV)$.

Let z_S be the projection to $\pi(C(V \oplus iV))\varphi_S^{1/2} = L^2(S)$ and similarly for z_T . Since z_S is in the commutant of the right representation of $C(V)$ on $L^2(S)$, it is approximated by the left multiplication of $C(V)$,

i.e., by elements in $\pi(C(V \oplus 0))$. Thus, if a unitary $U : L^2(S) \rightarrow L^2(T)$ intertwines π , then

$$U(\xi) = U(z_S \xi) = z_S U(\xi), \quad \xi \in L^2(S)$$

shows that $z_S = z_T$, i.e., $L^2(S) = L^2(T)$.

Otherwise, by the irreducibility of $\pi(C(V \oplus iV))$ on both $L^2(S)$ and $L^2(T)$,

$$\pi(C(V \oplus iV))\varphi_S^{1/2} \perp \pi(C(V \oplus iV))\varphi_T^{1/2},$$

i.e., $z_S \perp z_T$. Then $\Phi_S^{1/2}$ and $\Phi_T^{1/2}$ belong to inequivalent irreducible components of a representation of $C(V \oplus iV)$, whence they are orthogonal.

In either case, we have

$$(\Phi_S^{1/2} | \Phi_T^{1/2}) = \text{trace} \left(|\varphi_S^{1/2}\rangle \langle \varphi_S^{1/2}| |\varphi_T^{1/2}\rangle \langle \varphi_T^{1/2}| \right) = (\varphi_S^{1/2} | \varphi_T^{1/2})^2.$$

□

Remark 1. For factorial states, this kind of dichotomy is an immediate consequence of Schur's lemma. In the case of CAR, non-factorial quasi-free states are known to be decomposed into two pure states and we can work explicitly with these exceptional cases to get the dichotomy.

Remark 2. Let $C_0(V)$ be the even part of $C(V)$, which is the fixed point subalgebra by the parity automorphism. Let S be a covariance operator such that $S(I - S)$ is in the trace class and $\ker(2S - I)$ is even-dimensional. Then we can find Fock states φ_j ($j = 1, 2$) of $C(V)$ such that φ_j is quasi-equivalent to φ_S ($j = 1, 2$), the restrictions $\psi_j = \varphi_j|_{C_0(V)}$ are inequivalent pure states of $C_0(V)$, and $\varphi_S|_{C_0(V)} = (\psi_1 + \psi_2)/2$. Thus $\varphi_S|_{C_0(V)}$ is neither quasi-equivalent nor disjoint to both of ψ_j . See [7] for more information.

4. CCR DICHOTOMY

A state φ of a C^* -algebra C is said to be **standard** if $\overline{C\varphi^{1/2}} = \overline{\varphi^{1/2}C}$.

Example 4.1. Let $\varphi = \varphi_1 \otimes \varphi_2$ be a product state on $C = C_1 \otimes C_2$ with φ_1 a pure state of C_1 . Then $\overline{C_1\varphi_1^{1/2}C_1} \cong C_1\varphi_1^{1/2} \otimes \varphi_1^{1/2}C_1$ and φ is not standard if $\dim C_1\varphi_1^{1/2} = \dim \varphi_1^{1/2}C_1 \geq 2$.

Lemma 4.2.

- (i) Let S be the covariance form of a quasi-free state φ of a CCR C^* -algebra $C(V, \sigma)$. Then φ is standard if and only if the kernel of the ratio operator $\frac{S}{S+\bar{S}}$ on $V_S^{\mathbb{C}}$ is trivial. Here $V_S^{\mathbb{C}}$ denotes the Hilbert space induced from $S + \bar{S}$ on $V^{\mathbb{C}}$.

- (ii) Let S be the covariance operator of a quasi-free state φ of a CAR algebra $C(V)$. Then φ is standard if and only if $\ker S = \{0\}$.

Proof. Sufficiency: Since φ is a KMS-state, this follows from [15, Lemma 2.3].

Necessity: If a covariance form has a non-trivial kernel, the associated quasi-free state is factored through a pure state and therefore it is not standard in view of Example 4.1. \square

Corollary 4.3. Let φ be the quasi-free state of a covariance form S and suppose that the kernel of $S/(S + \overline{S})$ (CCR case) or the kernel of S (CAR case) is separable. Then we can find a standard quasi-free state φ' such that φ and φ' are quasi-equivalent.

Lemma 4.4. For standard states φ and ψ , $(\varphi^{1/2}|\psi^{1/2}) = 0$ if and only if $C\varphi^{1/2}C \perp C\psi^{1/2}C$, i.e., φ and ψ are disjoint.

Proof. This is a consequence of Schwarz inequality and the tracial property of the evaluation map in non-commutative L^p -theory: For $a, b \in C$,

$$\begin{aligned} |(a\varphi^{1/2}|b\psi^{1/2})| &= |(\psi^{1/4}b^*a\varphi^{1/4}|\psi^{1/4}\varphi^{1/4})| \\ &\leq \|\varphi^{1/4}a^*b\psi^{1/4}\|_2 \|\psi^{1/4}\varphi^{1/4}\|_2 \\ &= \|\varphi^{1/4}a^*b\psi^{1/4}\|_2 \sqrt{(\varphi^{1/2}|\psi^{1/2})} = 0. \end{aligned}$$

\square

Lemma 4.5. For quasi-free states φ and ψ , $(\varphi^{1/2}|\psi^{1/2}) > 0$ implies their quasi-equivalence, i.e., $C\varphi^{1/2}C = C\psi^{1/2}C$ ($C = C(V)$ or $C(V, \sigma)$).

Proof. We shall deal only with the case of CCR and the easier CAR case is omitted. In view of the determinant formula (Theorem 2.2), we first rewrite the condition that $(\varphi_S^{1/2}|\varphi_T^{1/2}) > 0$. The equivalence (i.e., mutual dominations) of $S + \overline{S}$ and $T + \overline{T}$ is necessary, which is assumed in the following. Because of

$$S + \overline{S} \leq 2A \leq 2(S + \overline{S}), \quad T + \overline{T} \leq 2B \leq 2(T + \overline{T}),$$

these as well as $A + B$ are equivalent. In particular, the ratio operator

$$\frac{\sqrt{AB}}{A + B}$$

is invertible and the transition probability does not vanish if and only if

$$I - 2\frac{\sqrt{AB}}{A + B} = \left(\sqrt{\frac{A}{A + B}} - \sqrt{\frac{B}{A + B}} \right)^2$$

is in the trace-class. In view of

$$\left(\sqrt{\frac{A}{A+B}} + \sqrt{\frac{B}{A+B}} \right)^2 \left(\sqrt{\frac{A}{A+B}} - \sqrt{\frac{B}{A+B}} \right)^2 = \left(\frac{A}{A+B} - \frac{B}{A+B} \right)^2$$

and the invertibility of $\frac{A}{A+B}$ and $\frac{B}{A+B}$, the condition is equivalent to requiring that

$$\frac{2A}{A+B} - \frac{2B}{A+B} = \frac{S + \bar{S} + 2\sqrt{S\bar{S}}}{A+B} - \frac{T + \bar{T} + 2\sqrt{T\bar{T}}}{A+B}$$

is in the Hilbert-Schmidt class. The last condition is equivalent to the quasi-equivalence of φ_S and φ_T by [3, Theorem, Proposition 6.6, Proposition 9.1].

In this way, we have proved that φ_S and φ_T are quasi-equivalent if $(\varphi_S^{1/2} | \varphi_T^{1/2}) > 0$. \square

In the case of CAR algebras, the converse of Lemma 3.7 is false.

Proposition 4.6. Let S and T be covariance operators on $V^{\mathbb{C}}$ with P and Q their quadratures on $(V \oplus iV)^{\mathbb{C}} = V^{\mathbb{C}} \oplus V^{\mathbb{C}}$. Assume that φ_S and φ_T are quasi-equivalent. Then $(\varphi_S^{1/2} | \varphi_T^{1/2}) = 0$ if and only if $P \wedge (I - Q) \neq 0$.

In the case of CCR C^* -algebras, however, the transition probability is already sensitive to the dichotomy:

Theorem 4.7 (Dichotomy). Let (V, σ) be a presymplectic vector space (σ being an alternating bilinear form on V) and S, T be covariance forms with the associated quasi-free states φ_S, φ_T . Then the following conditions are equivalent.

- (i) Two states φ_S and φ_T are quasi-equivalent.
- (ii) The transition probability $(\varphi_S^{1/2} | \varphi_T^{1/2})$ is strictly positive.
- (iii) Positive forms $(\sqrt{S} + \sqrt{\bar{S}})^2$ and $(\sqrt{T} + \sqrt{\bar{T}})^2$ on $V^{\mathbb{C}}$ are Hilbert-Schmidt equivalent.

Otherwise, φ_S and φ_T are disjoint.

Proof. In view of Lemma 4.5, it suffices to show that the condition $(\varphi_S^{1/2} | \varphi_T^{1/2}) = 0$ implies the disjointness of φ_S and φ_T .

By replacing $V^{\mathbb{C}}$ with $V_{A+B}^{\mathbb{C}}$, we may assume that $A + B$ is non-degenerate and complete on $V^{\mathbb{C}}$. If

$$\ker \left(\sqrt{\frac{A}{A+B}} \sqrt{\frac{B}{A+B}} \right)$$

is not trivial, we can find $0 \neq h \in V^{\mathbb{C}}$ such that $(A/(A+B))h = 0$ or $(B/(A+B))h = 0$. We may assume that the former is the case. Since the operator $A/(A+B)$ is self-conjugate, we can further assume that $h = \bar{h}$, i.e., $h \in V$. Now the condition $A(h, h) = 0$ implies $S(h, h) = 0 = \bar{S}(h, h)$, whence $S(h, v) = 0 = \bar{S}(h, v)$ and $\sigma(h, v) = 0$ for any $v \in V$. Thus $\{e^{ith}\}_{t \in \mathbb{R}}$ is in the center of $C^*(V, \sigma)$. Since $h \neq 0$ with respect to the inner product $A+B$, we see $B(h, h) \neq 0$ and therefore $T(h, h) \neq 0$.

We now compare the spectral decomposition of $\{e^{ith}\}$ when represented by left multiplication: On the subspace $C^*(V, \sigma)\varphi_S^{1/2}C^*(V, \sigma)$, it is represented by the identity operator, whereas on the subspace $\mathcal{H} = C^*(V, \sigma)\varphi_T^{1/2}C^*(V, \sigma)$ it is isomorphic to a direct sum of the multiplication operator $\{e^{itx}\}$ on $L^2(\mathbb{R}, \gamma)$ (γ being a Gaussian measure);

$$(e^{itx}\xi)(\tau) = e^{it\tau}\xi(\tau) \quad \text{with} \quad \xi \in L^2(\mathbb{R}, \gamma).$$

Thus φ_S and φ_T are disjoint.

We now assume that the kernel of $\sqrt{AB}/(A+B)$ is trivial. Then, by the determinant formula for the transition probability, $(\varphi_S^{1/2}|\varphi_T^{1/2}) = 0$ implies that the bounded operator

$$\left(\sqrt{\frac{A}{A+B}} - \sqrt{\frac{B}{A+B}} \right)^2$$

is not in the trace class. In particular, we can find a sequence $\{h_j\}_{j \geq 1}$ of $(A+B)$ -orthonormal vectors in $V^{\mathbb{C}}$ such that

$$\sum_j (A+B)(h_j, \left(\sqrt{\frac{A}{A+B}} - \sqrt{\frac{B}{A+B}} \right)^2 h_j) = +\infty.$$

Let M be the set of monomials of $S/(A+B)$, $T/(A+B)$, $\bar{S}/(A+B)$ and $\bar{T}/(A+B)$. Let $W^{\mathbb{C}}$ be the closed subspace spanned by $\{Mh_j, M\bar{h}_j\}_{j \geq 1}$. Since M is countable, $W^{\mathbb{C}}$ is separable as a hilbertian vector space. Clearly $W^{\mathbb{C}}$ is invariant under four generators of M . In view of $i\sigma = S - \bar{S}$, $W^{\mathbb{C}}$ is also invariant under $\sigma/(A+B)$. Since M is closed under taking conjugate, so is $W^{\mathbb{C}}$, which justifies the notation, i.e., W denotes the real part of $W^{\mathbb{C}}$. Let W^{\perp} be the orthogonal complement of W relative to $A+B$ so that $(V, \sigma) = (W, \sigma) \oplus (W^{\perp}, \sigma)$ with S and T diagonally decomposed. Let S_W and T_W be the reduced covariance forms. Then

$$\frac{\sqrt{A_W B_W}}{A_W + B_W}$$

is the restriction of $\sqrt{AB}/(A+B)$ to the subspace $W^{\mathbb{C}}$ and

$$\begin{aligned} & \text{trace} \left(\sqrt{\frac{A_W}{A_W + B_W}} - \sqrt{\frac{B_W}{A_W + B_W}} \right)^2 \\ &= \text{trace}_{W^{\mathbb{C}}} \left(\sqrt{\frac{A}{A+B}} - \sqrt{\frac{B}{A+B}} \right)^2 \\ &\geq \sum_j (A+B)(h_j, \left(\sqrt{\frac{A}{A+B}} - \sqrt{\frac{B}{A+B}} \right)^2 h_j) = +\infty, \end{aligned}$$

which means that φ_{S_W} and φ_{T_W} are not quasi-equivalent by Theorem 2.1.

Now the obvious identification

$$\overline{C^*(V, \sigma) \varphi_S^{1/2} C^*(V, \sigma)} = \overline{C^*(W, \sigma) \varphi_{S_W}^{1/2} C^*(V, \sigma) \otimes C^*(W^\perp, \sigma) \varphi_{S_{W^\perp}}^{1/2} C^*(W^\perp, \sigma)}$$

reveals that the disjointness of φ_S and φ_T follows from that of φ_{S_W} and φ_{T_W} . Thus the problem is reduced to the case $W = V$ so that V is separable relative to the inner product $A+B$ (so we omit the suffix W) and that φ_S and φ_T are not quasi-equivalent. We can then find standard covariance forms S' and T' such that φ_S and $\varphi_{S'}$ (resp. φ_T and $\varphi_{T'}$) are quasi-equivalent by Corollary 4.3.

Since φ_S and φ_T are not quasi-equivalent, the same holds for $\varphi_{S'}$ and $\varphi_{T'}$, which implies the disjointness of $\varphi_{S'}$ and $\varphi_{T'}$ by Lemma 4.4. Thus $L^2(S) = L^2(S')$ is orthogonal to $L^2(T) = L^2(T')$, proving the disjointness of φ_S and φ_T . \square

REFERENCES

- [1] P.M. Alberti and A. Uhlmann, On Bures distance and *-algebraic transition probability between inner derived positive linear forms over W^* -algebras, *Acta Appl. Math.*, 60(2000), 1–37.
- [2] H. Araki, On quasifree states of CAR and Bogoliubov automorphisms, *Publ. RIMS*, 6(1970/1971), 385–442.
- [3] H. Araki and S. Yamagami, On quasi-equivalence of quasifree states of the canonical commutation relations, *Publ. RIMS*, 18(1982), 703–758.
- [4] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics II*, Springer-Verlag, 1979.
- [5] U. Haagerup, L^p -spaces associated with an arbitrary von Neumann algebra, *Colloques internationaux du CNRS*, No. 274, 1977.
- [6] S. Kakutani, On equivalence of infinite product measures, *Ann. Math.*, 49(1948), 214–224.

- [7] T. Matsui, Factoriality and quasi-equivalence of quasifree states for Z_2 and $U(1)$ invariant CAR algebras, *Rev. Roumaine Math. Pures Appl.*, 32(1987), 693-700.
- [8] R.T. Power and E. Størmer, Free states of the canonical anticommutation relations, *Commun. Math. Phys.*, 16(1970), 1-33.
- [9] W. Pusz and S.L. Woronowicz, Functional calculus for sesquilinear forms and the purification map, *Rep. Math. Phys.*, 8(1975), 159-170.
- [10] G.A. Raggio, Comparison of Uhlmann's transition probability with the one induced by the natural cone of von Neumann algebras in standard form, *Lett. Math. Phys.*, 6(1982), 233-236.
- [11] D. Shale, Linear symmetries of free Boson fields, *Trans. Amer. Math. Soc.*, 103(1962), 149-167.
- [12] D. Shale and W.F. Stinespring, States on the Clifford algebra, *Ann. Math.*, 80(1964), 365-381.
- [13] A. Van Daele, Quas-equivalence of quasi-free states on the Weyl algebra, *Commun. Math. Phys.*, 21(1971), 171-191.
- [14] S. Yamagami, Algebraic aspects in modular theory, *Publ. RIMS*, 28(1992), 1075-1106.
- [15] S. Yamagami, Geometric mean of states and transition amplitudes, *Lett. Math. Phys.*, 84(2008), 123-137.
- [16] S. Yamagami, Geometry of quasi-free states of CCR-algebras, *Int. J. Math.*, 21(2010), 875-913.

FACULTY OF MATHEMATICS, KYUSHU UNIVERSITY

E-mail address: matsui@math.kyushu-u.ac.jp

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY

E-mail address: yamagami@math.nagoya-u.ac.jp

URL: <http://www.math.nagoya-u.ac.jp/~yamagami/>